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Fixed point set characterizations of Peano continua and absolute retracts

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Abstract

We extend the notion of *absolute fixed point sets* to the setting of continuum-valued maps whose point images have small diameters. We demonstrate that the resulting class of spaces (ε-MAFS) coincides with the class of absolute fixed point sets and with the class of absolute retracts in the one-dimensional and planar settings, but that the class of n-dimensional absolute fixed point sets is strictly contained in ε-MAFS for all $n > 1$. Moreover, we will give two characterizations for the class of Peano continua in terms of ε-MAFS.

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1. Introduction

A *compactum* is a nonempty, compact metric space, and a *continuum* is a connected compactum. A *Peano continuum* is a locally connected continuum. If Z is a compactum, then $C(Z)$ (2^Z) denotes the space of all subcontinua (subcompacta) of Z together with the Hausdorff metric. We will write $\langle Z, \rho\rangle$ to denote a compactum, Z, with metric, ρ.

A *map* or *mapping* is a continuous function, and a map into $C(Z)$ (2^Z) for some compactum, Z, will be referred to as a *continuum-valued map* (*multi-valued map*). A *fixed point of a multi-valued map*, F, is a point, p, such that $p \in F(p)$; the *fixed point set of* F is the set of all fixed points of F, and is denoted by $\mathcal{F}(F)$. We want to be able to talk about a fixed point set of a compactum, but there could easily be confusion about the type of map involved. Hence, for a compactum, Z, we use the phrase *multi-valued fixed point set of* Z to refer to a subset, A, of Z such that A is the fixed point set of some multi-valued

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map $F : Z \to 2^Z$. In case that F is continuum-valued, we call A a *continuum-valued fixed point set of* Z. Following [2], we will use AR and ANR to denote the classes of absolute retracts and absolute neighborhood retracts, and we will use ANR($\mathcal{M}$) to denote the class of absolute neighborhood retracts that are not necessarily compact.

In [5], John Martin introduces the notion of absolute fixed point sets in the setting of single-valued maps as follows: A compactum, A, is an *absolute fixed point set* provided that whenever A is embedded as a subspace, A', of any compactum, Z, then A' is the fixed point set of Z for some $f : Z \to Z$. We use AFS to denote the class of all (compact) absolute fixed point sets in the setting of single-valued maps. In [7], Martin proves that every member of AFS is both contractible and locally contractible. The following corollaries follow from this result:

(i) a finite-dimensional compactum is a member of AFS if and only if it is an absolute retract, and
(ii) a compactum is an absolute retract if and only if it belongs to ANR $\cap$ AFS.

The notion of absolute fixed point sets was extended in [9] to the setting of multi-valued mappings. A *multi-valued absolute fixed point set* is a compactum, A, such that whenever A is embedded as a subspace, A', of any compactum, Z, then A' is a continuum-valued fixed point set of Z. We will use MAFS to denote the class of all (compact) multi-valued absolute fixed point sets. In [9], MAFS is characterized as the class of those compacta with only finitely many components, all but at most one of which are locally connected. A further generalization of MAFS was defined and characterized in [8].

This paper extends Martin's work by considering AFS in the setting of continuum-valued maps whose point images have small diameters. The following definition will be convenient:

Definition 1.1. Let Z be a compactum with metric, ρ. For any $\varepsilon \geqslant 0$, let

$$C_{\rho,\varepsilon}(Z) = \bigl\{A \in C(Z)\colon \operatorname{diam}_\rho(A) \leqslant \varepsilon\bigr\},$$

where $\operatorname{diam}_\rho(A) = \sup\{\rho(x, y)\colon x, y \in A\}$. We will write $C_\varepsilon(Z)$ when the metric on Z is understood.

We remark that $C_0(Z)$ is homeomorphic to Z, and that if $\varepsilon \geqslant \operatorname{diam}(K)$ for every component, K, of Z, then $C_\varepsilon(Z) = C(Z)$. Furthermore, $C_\varepsilon(Z)$ is always a compactum, and $C_\varepsilon(Z)$ is a continuum if Z is connected.

Proposition 1.2. *Let $\langle Z, \rho\rangle$ be a compactum, and let d be a metric for Z that is equivalent to ρ. Then, for any $\delta > 0$, there exists some $\gamma > 0$ such that $C_{d,\gamma}(Z) \subseteq C_{\rho,\delta}(Z)$.*

Proof. Use compactness to cover Z with finitely many open sets $U_1, U_2, \dots, U_n$ such that $\operatorname{diam}_\rho(U_i) \leqslant \delta$ for each i. By the Lesbesgue covering lemma [13, 22.5], there exists some $\gamma > 0$ such that if E is any subset of Z for which $\operatorname{diam}_d(E) < 2\gamma$, then $E \subseteq U_i$ for some i. Thus, since $\operatorname{diam}_d(E) < 2\gamma$ for every $E \in C_{d,\gamma}(Z)$, it follows that $C_{d,\gamma}(Z) \subseteq C_{\rho,\delta}(Z)$. □

We are now prepared to define the subclasses of MAFS that will be the subject of this paper:

Definition 1.3. We say that a compactum, A, is an ε-MAFS (respectively, 0-MAFS) provided that whenever A is embedded as a subspace, A', of any compactum, Z, then for every δ with $\delta > 0$ (respectively, $\delta = 0$), there is some map, $F_\delta : Z \to C_\delta(Z)$, for which $\mathcal{F}(F_\delta) = A'$.

Definition 1.4. We say that a compactum, A, is a *weak* ε-MAFS (respectively, *weak* 0-MAFS) provided that whenever A is embedded as a subspace, A', of any compactum, Z, then for every δ with $\delta > 0$ (respectively, $\delta = 0$), there is some map, $F_\delta : Z \to C(Z)$, that satisfies the following conditions:

$$\mathcal{F}(F_\delta) = A', \tag{1}$$

$$F_\delta(z) \in C_\delta(Z) \quad \text{for every } z \in A'. \tag{2}$$

We remark that the definitions of (weak) ε-MAFS and (weak) 0-MAFS do not depend upon the metric considered on the compactum, Z. This independence follows easily from (1.2).

Notation 1.5. We will use AFS, ε-MAFS, etc. to refer to classes as well as to the members within those classes. When the context does not make clear to which of these we refer, we will use AFS-space, ε-MAFS-space, etc. to denote the class members.

Remark 1.6. The relationships in the following diagram are easily verified:

$$\begin{array}{ccccc} \mathrm{AR} \subset \mathrm{AFS} = & 0\text{-}\mathrm{MAFS} & \subset & \varepsilon\text{-}\mathrm{MAFS} \\ & \cap & & \cap \\ & \text{weak } 0\text{-}\mathrm{MAFS} & \subset & \text{weak } \varepsilon\text{-}\mathrm{MAFS} \end{array}$$

In this paper, we will show that the containment in the bottom row of (1.6) is reversible, while all of the other indicated containments are strict. We will also show that the containments given in the top row of (1.6) are reversible in the settings of one-dimensional and planar continua. The main results of the paper are stated below:

Theorem 1.7. *The following statements are equivalent for any compactum, A*:

(1) *A is a Peano continuum*;
(2) *A is a weak* 0*-MAFS-space*;
(3) *A is a weak ε-MAFS-space.*

Theorem 1.8. *If A is any one-dimensional continuum or any planar continuum, then the following statements are equivalent*:

(1) *A is an absolute retract*;
(2) *A is an* AFS-*space*;
(3) *A is an ε-MAFS-space.*

2. Proof of Theorem 1.7

Proposition 2.1. *Every weak ε-MAFS is connected.*

Proof. Let A be a compactum containing nonempty, disjoint, closed subsets, E_1 and E_2, such that $A = E_1 \cup E_2$. Let $Y = J_1 \cup W_1 \cup W_2 \cup J_2$, where

$$W_1 = \left\{\left(x, \sin\frac{1}{x}\right):\ x \in (0,1]\right\},$$
$$W_2 = \left\{(2-x, y):\ (x,y) \in W_1\right\},$$
$$J_1 = \left\{(0,y):\ y \in [-1,1]\right\},$$
$$J_2 = \left\{(2,y):\ y \in [-1,1]\right\}.$$

Without loss of generality, we may assume that A and Y are disjoint. Define Z to be the compactum formed by identifying $(0,-1)$ with a point of E_1, and $(2,-1)$ with a point of E_2; then, let d be any metric on Z which gives the quotient topology induced on Z by this identification. Let p_1, p_2, q_1 and q_2 denote those points of Z which correspond to $(0,-1)$, $(2,-1)$, $(0,1)$ and $(2,1)$, respectively. Let δ be any nonnegative number such that $\delta < \min\{d(p_1,q_1), d(p_2,q_2)\}$. We will use W_i, J_i, A and Y to denote the natural copies of these spaces in Z. We wish to show that

($*$) there is no continuum-valued map $F : Z \to C(Z)$ for which $\mathcal{F}(F) = A$ and such that $F(z) \in C_\delta(Z)$ for every $z \in A$.

Suppose that such a mapping does exist. Then, by the continuity of F and the assumption that no point of $W_1 \cup W_2$ is fixed by F, we may assume without loss of generality that

$$z \text{ lies to the left of } F(z) \quad \text{for every } z \in W_1 \cup W_2. \tag{3}$$

Observe that since $p_1 \in F(p_1)$ and $\operatorname{diam}(p_1) < \delta \leqslant d(p_1,q_1)$, it follows that $q_1 \notin F(p_1)$. Suppose that there is some $z \in J_1 - \{p_1\}$ such that $F(z)$ shares a point of J_1 above z. Then, since $F(z)$ is connected and $z \notin F(z)$, we would have that all of $F(z)$ must lie in J_1 above z. Moreover, every member of J_1 above z would have this property by the continuity of F. It would follow that $F(q_1) = \{q_1\}$, contradicting the assumption that $\mathcal{F}(F) = A$. Therefore, we must have that $q_1 \notin F(z)$ for every $z \in J_1$. So, by the continuity of F, there exists an open set, U_z, about each $z \in J_1$ such that $q_1 \notin F(y)$ for every $y \in U_z$. Thus, by the compactness of J_1, it follows that there exists a subcontinuum, K, of Y such that $J_1 \subset K$, $K \cap W_1 \neq \emptyset$, and for which

$$q_1 \notin F(y) \quad \text{for every } y \in K. \tag{4}$$

Let $M = \bigcup\{F(y)\colon\ y \in K\}$. Then, M is a continuum by [10, 1.49], M contains p_1 since p_1 is fixed by F, and M contains points of $Y - J_1$ by (3), since $K \cap W_1 \neq \emptyset$. It follows that $J_1 \subset M$. Hence, $q_1 \in F(y)$ for some $y \in K$, contrary to (4). This verifies $(*)$, as desired. $\square$

Proposition 2.2. *Every weak ε-MAFS is locally connected.*

Proof. We will show that if A is a compactum that is not locally connected, then A fails to be a weak ε-MAFS. By (2.1), we may assume that A is a continuum. Therefore, since A is not locally connected, there exists some $a_0 \in A$ at which A fails to be connected *im kleinen* [11, 5.22b]. Since A fails to be connected im kleinen at a_0, there exists some open set, V, about a_0 which contains a sequence of points, $\{a_i\}_{i=1}^{\infty}$, and a sequence of subsets, $\{C_i\}_{i=0}^{\infty}$, such that

$$a_0 = \lim a_i, \tag{5}$$

$$C_i \text{ is the component of } a_i \text{ in } V \quad \text{for each } i \geqslant 0, \tag{6}$$

$$C_i \cap C_j = \emptyset \quad \text{for every } i \neq j. \tag{7}$$

Let $S = \{(x, \sin\frac{1}{x})\colon\ 0 < x \leqslant 1\}$, and let I_j denote the interval $[0, \frac{1}{j}]$ for each $j \in \{1, 2, \ldots\}$. For each $i \in \{1, 2, \ldots\}$ and each $j \in \{1, 2, \ldots, i\}$, let $f_{i,j}\colon S \to \mathbb{R}^3$ be an embedding for which

$$\overline{f_{i,j}(S)} - f_{i,j}(S) = \{0\} \times \{0\} \times \left[0, \frac{1}{j}\right], \tag{8}$$

$$f_{i,j}(S) \subset \left\{(x, y)\colon\ x^2 + y^2 \leqslant \frac{1}{k}\right\} \times I_j, \quad \text{where } k = \frac{i(i-1)}{2} + j, \tag{9}$$

$$f_{i,j}(S) \cap f_{i',j'}(S) \neq \emptyset \quad \text{if and only if} \quad i = i' \text{ and } j = j'. \tag{10}$$

In other words, $\{f_{i,j}(S)\}$ is a sequence of $\sin\frac{1}{x}$ curves with decreasing widths and alternating heights: The first term of this sequence, $f_{1,1}(S)$, fits inside the tube centered on the z-axis with radius 1, while the fifth term, $f_{3,2}(S)$ $\bigl(5 = \frac{3(3-1)}{2} + 2\bigr)$, fits inside the tube centered on the z-axis with radius $\frac{1}{5}$. Moreover, $f_{1,1}(S)$ has height 1, and the heights of $f_{3,1}(S)$, $f_{3,2}(S)$, $f_{3,3}(S)$ and $f_{4,1}(S)$ have heights 1, $\frac{1}{2}$, $\frac{1}{3}$ and 1, respectively. Define $p = (0, 0, 0)$ and $q_j = (0, 0, \frac{1}{j})$ for each $j \in \{1, 2, \ldots\}$. For each $i \in \{1, 2, \ldots\}$ and each $j \in \{1, 2, \ldots, i\}$, let $p_{i,j}$ be a point of $f_{i,j}(S)$ such that $p = \lim p_{i,j}$. Define $Y = \bigcup_{i<\infty,\ j\leqslant i} \overline{f_{i,j}(S)}$. Without loss of generality, we may assume that A and Y are disjoint. Form Z from A and Y by identifying a_0 with p, and $p_{i,j}$ with a_{2k} for each i and j, where $j \leqslant i$ and $k = \frac{i(i-1)}{2} + j$; that is, identify $p_{1,1}$ with a_2, $p_{2,1}$ with a_4, $p_{2,2}$ with a_6, $p_{3,1}$ with a_8, and so on. Then, Z is a continuum which contains (a copy of) A. We will use A, Y, C_i, $f_{i,j}(S)$, etc. to denote the natural copies of these spaces in Z. Let d be any metric on Z which gives the quotient topology induced on Z by this identification. Pick any $\delta \geqslant 0$ such that $\delta < \frac{1}{2}d(a_0, A - V)$. We wish to show that

$(*)$ there is no continuum-valued map $F\colon Z \to C(Z)$ for which $\mathcal{F}(F) = A$ and such that $F(z) \in C_\delta(Z)$ for every $z \in A$.

Suppose that such a mapping does exist. It follows that $F(p) \cap Y = \{p\}$, for otherwise we would have $F(a_{2i-1}) \cap Y \neq \emptyset$ for all sufficiently large i; however, since a_{2i-1} is fixed by F, this would imply that $\operatorname{diam}[F(a_{2i-1})] > \frac{1}{2}d(a_0, A - V) > \delta$ by (5), (6), (7) and the assumption that $Y \cap A = \{a_0, a_2, a_4, a_6, \ldots\}$. It now follows by an argument similar to that given in 2.1 that

$$\text{if } z_1 \text{ lies above } z_2 \text{ in } I_1, \quad \text{then } z_1 \notin F(z_2). \tag{11}$$

Since $\operatorname{diam}[F(p)] \leqslant \delta$, the continuity of F guarantees the existence of an open subset, W, of Z such that $p \in W$ and

$$F(z) \cap A \subset V \quad \text{for every } z \in W. \tag{12}$$

Observe that since $\operatorname{diam}(I_j) \to 0$ as $j \to \infty$, W contains I_{j_0} for some $j_0 \in \{1, 2, \ldots\}$. Thus, (8) and (9) give that for some $N_1 < \infty$, W contains $\overline{f_{i,j_0}(S)}$ for all $i \geqslant N_1$. By (5), (6) and our choice of δ, there exists some $N_2 < \infty$ such that

$$\text{for all } i \geqslant N_2, \quad \operatorname{diam}(C_{2i}) > \frac{1}{2}d(a_0, A - V) > \delta. \tag{13}$$

Let $N = \max\{N_1, N_2\}$, and let $K = \overline{\bigcup_{i \geqslant N} f_{i,j_0}(S)}$. Now, by (11) we can find an open set, U_z, about each $z \in I_{j_0}$ such that $q_{j_0} \notin F(y)$ for all $y \in U_z$. Using the compactness of I_{j_0}, we can find a subcontinuum, M, of K such that $I_{j_0} \subset M$, $f_{i_0,j_0}(S) \subset M$ for some $i_0 > N$, and

$$q_{j_0} \notin F(y) \quad \text{for all } y \in M. \tag{14}$$

Let $M' = \bigcup\{F(y)\colon\ y \in M\}$. Then M' is a continuum [10, 1.49] that contains both p and p_{i_0,j_0}. Since $p_{i_0,j_0} = a_{2i}$ for some i, we have by (12), (13) and (14) that there is no connected subset of $M' \cap A$ that contains both p and p_{i_0,j_0}. So, by (10), M' must contain a connected subset of $\overline{f_{i_0,j_0}(S)}$ that contains both p and p_{i_0,j_0}; thus, M' must contain q_{j_0}, contradicting (14). This verifies ($*$), and completes the proof of the proposition. □

Proposition 2.3. *Every Peano continuum is a weak* 0*-MAFS.*

Proof. Let Z be a compactum, and let A be any locally connected subcontinuum of Z. Define $G: A \to C(A)$ by $G(z) = \{z\}$ for all $z \in A$. Because $C(A)$ is an absolute retract [10, 1.96], we can extend G to a map $F: Z \to C(A)$. Clearly, we have that $\mathcal{F}(F) = A$ and that $\operatorname{diam}[F(z)] = 0$ for all $z \in A$. This proves 2.3. □

We proved in 2.3 that every Peano continuum is a weak 0-MAFS, and we observed in 1.6 that every weak 0-MAFS is a weak ε-MAFS. Finally, every weak ε-MAFS is a Peano continuum by 2.1 and 2.2. This completes the proof of 1.7.

3. Proof of Theorem 1.8

We have just proven that the classes of weak 0-MAFS and weak ε-MAFS are equivalent. Despite this fact, we will later demonstrate that there are examples of n-dimensional ε-MAFS-spaces which fail to belong to 0-MAFS; that is, ε-MAFS-spaces which are not

AFS-spaces. Nonetheless, we will prove in this section that ε-MAFS and AFS are each equivalent to the class of absolute retracts in the 1-dimensional and planar settings.

In the proofs which follow, we use S^n to denote the n-dimensional unit sphere, and B^{n+1} to denote the convex hull of S^n. Recall that a mapping between two spaces is said to be *inessential* provided that it is homotopic to a constant map.

Lemma 3.1. *If $f : S^n \to S^n$ is inessential, then $f(x) = -x$ for some $x \in S^n$.*

Proof. Since $f : S^n \to S^n$ is inessential, it fails to be homotopic to the identity map on S^n. Hence, the degree of f is different than 1 [3, p. 339; Example 2], [3, p. 352; Theorem 7.4]. Therefore, $f(x) = -x$ for some $x \in S^n$ [3, p. 354; Problem 7.1]. $\square$

Proposition 3.2. *S^n fails to be an ε-MAFS for every $n \in \{1, 2, \ldots\}$.*

Proof. Let W be a homeomorphic copy of $B^{n+1} - S^n$ which has been deformed in $\sin \frac{1}{x}$ fashion; specifically, let

$$W = \left\{ \left(x, \sin \frac{1}{1 - \|x\|} \right) \colon x \in B^{n+1} - S^n \right\},$$

where $\|x\|$ denotes the distance between x and the origin. Let I_1 be an arc in W, and let I_2 be an arc in $S^n \times \{-1\}$. Then, let H be a homeomorphic copy of the open interval $(0, 1)$ such that $\overline{H} \cap \overline{W} = I_1 \cup I_2$. Define $Z = \overline{W} \cup \overline{H}$.

Now, let $A = S^n \times \{-1\}$. We will prove 3.2 by showing that A is not the fixed point set of any $F : Z \to C_\varepsilon(Z)$ whenever $\varepsilon < \min\{\operatorname{diam}(I_1), \operatorname{diam}(I_2), \frac{1}{2}\}$. For purpose of contradiction, suppose that some $F : Z \to C_\varepsilon(Z)$ does exist with $\mathcal{F}(F) = A$. By our choice of ε, it is easy to see that $C_\varepsilon(W)$, $C_\varepsilon(\overline{W} - W)$ and $C_\varepsilon(H)$ are the only arc components (maximal arcwise connected subsets) of $C_\varepsilon(Z)$. Therefore, since W is arcwise connected, it follows that

$$F(W) \quad \text{is contained in one of } C_\varepsilon(W), C_\varepsilon(\overline{W} - W), \text{ and } C_\varepsilon(H). \tag{15}$$

Suppose that $F(W) \subseteq C_\varepsilon(\overline{W} - W)$. Then, since each $p \in A$ is fixed by F and $\operatorname{diam}(F(p)) \leqslant \varepsilon$, it follows by the uniform continuity of F that there is some $r \in (0, 1)$ with $\sin \frac{1}{1-r} = -1$ for which

$$H_d\bigl(\{q\}, F(q)\bigr) < 2\varepsilon \tag{16}$$

for every $q \in W$ with $\|q\| = r$. Let ρ_1 denote the natural retraction mapping from $C_\varepsilon(\overline{W} - W)$ onto $C_\varepsilon(A)$, and observe that

$$H_d\bigl(\{q\}, (\rho_1 \circ F)(q)\bigr) \leqslant H_d\bigl(\{q\}, F(q)\bigr) \tag{17}$$

for every $q \in W$ with $\|q\| = r$. Define a map, $\mu : C_\varepsilon(A) \to A$, as follows: For each $E \in C_\varepsilon(A)$, let D_E denote the smallest convex subset of B^{n+1} that contains E, and for which the base of D_E is an n-dimensional disk. Let R denote the ray in $\mathbb{R}^{n+1}$ with endpoint $(0, 0, \ldots, -1)$ which passes through the center of this disk, and let $\mu(E) = S^n \cap R$. Then, μ is a continuous function such that

$$H_d\bigl(E, \bigl\{\mu(E)\bigr\}\bigr) \leqslant \operatorname{diam}(E) < \varepsilon \tag{18}$$

for every $E \in C_\varepsilon(E)$. Finally, let $S' = \{(x, -1)\colon\ x \in B^{n+1}, \|x\| = r\}$, and let $\rho_2 : A \to S'$ denote the radial retraction map. Define $D' = \{(x, \sin\frac{1}{1-\|x\|})\colon\ \|x\| \leqslant r\}$, and observe that D' is an $(n+1)$-dimensional disk whose boundary is S'. Define a map $g : D' \to S'$ by

$$g = \rho_2 \circ \mu \circ \rho_1 \circ F : D' \to S'.$$

Note that $g\,|_{S'} : S' \to S'$ shares no point in common with the antipodal map on S' by (16), (17) and (18). Therefore, we have by 3.1 that $g \mid S'$ is essential. It follows from [11, 12.35] that g is essential. However, since D' is homeomorphic to B^{n+1}, it follows that g is inessential. Therefore, $F(W) \nsubseteq C_\varepsilon(\overline{W} - W)$.

Next, observe that if $F(W) \subseteq C_\varepsilon(H)$, it would follow that $F(A) \subseteq \overline{C_\varepsilon(H)}$ since $A \subseteq \overline{W}$. Since this would contradict the assumption that the points of A are fixed by F, we must conclude from (15) that

$$F(W) \subseteq C_\varepsilon(W). \tag{19}$$

Now note that if $F(H) \subseteq C_\varepsilon(W)$, then, since $I_2 \subseteq \mathcal{F}(F)$, there must be some connected subset of $C_\varepsilon(W)$ whose remainder intersects $C_\varepsilon(I_2)$. However, any subset of $C_\varepsilon(W)$ whose closure meets $C_\varepsilon(I_2)$ must contain points in infinitely many "troughs" of the $\sin\frac{1}{x}$ wave. Thus, if the subset is connected, it must also contain points at all heights of the wave. Therefore, since $\varepsilon < \frac{1}{2}$, no connected subset of $C_\varepsilon(W)$ exists whose closure meets $C_\varepsilon(I_2)$. Moreover, if $F(H) \subseteq C_\varepsilon(\overline{W} - W)$, then we must have $F(I_1) \subseteq C_\varepsilon(\overline{W} - W)$; hence, $F(W) \subseteq C_\varepsilon(\overline{W} - W)$, contrary to (19). It follows that

$$F(H) \subseteq C_\varepsilon(H). \tag{20}$$

Since $\overline{W} \cap \overline{H} = I_1 \cup I_2$, it follows from (19) and (20) that $F(I_1) \subseteq C_\varepsilon(I_1)$. Therefore, it follows from [12] that $I_1 \cap \mathcal{F}(F) \neq \emptyset$, contrary to our assumption that $\mathcal{F}(F) = A$. This completes the proof that S^n is not an ε-MAFS. □

Proposition 3.3. *Every retract of an ε-MAFS is an ε-MAFS.*

Proof. Let Y be an ε-MAFS, and let A be a retract of Y. Let Z be a compactum containing a copy, A', of A. Let h be a homeomorphism from A onto A'. Now, assuming as we may that $Z \cap Y = \emptyset$, form the attaching space $W = Z \cup_h Y$ [11, 3.18] in which we consider Z, Y and $A = A'$ as subspaces in the natural way. By assumption, there exists a retraction, r, from Y onto $A = A'$. Let $\bar{r}$ be the retraction from W onto Z given by

$$\bar{r}(z) = \begin{cases} z & \text{if } z \in Z, \\ r(z) & \text{if } z \in Y. \end{cases}$$

Since W is a compactum [11, 3.19], we have that $\bar{r}$ is uniformly continuous. Hence, for every $\delta > 0$, there is some $0 < \delta' \leqslant \delta$ such that $\bar{r}(E) \in C_\delta(W)$ for every $E \in C_{\delta'}(W)$. Since Y is an ε-MAFS, there exists some map, $G : W \to C_{\delta'}(W)$ with $\mathcal{F}(G) = Y$. Define $F : Z \to C_\delta(Z)$ by $F(z) = \bar{r}[G(z)]$ for each $z \in Z$. Then, F is a map with $\mathcal{F}(F) = A$. Therefore, A is an ε-MAFS. This proves 3.3. □

Proposition 3.4. *Every one-dimensional ε-MAFS is an absolute retract.*

Proof. Let A be a one-dimensional ε-MAFS. Then, by 3.2 and 3.3, A contains no simple closed curve as a retract. Since A is one-dimensional, it follows that A contains no simple closed curve. Therefore, since A is a Peano continuum by 2.1 and 2.2, we have that A is a dendrite. It follows [4, p. 339] that A is an absolute retract. □

Proposition 3.5. *Every ε-MAFS in $\mathbb{R}^2$ is an absolute retract.*

Proof. Let $A \subset \mathbb{R}^2$ be an ε-MAFS. Then, by 2.1, 2.2, 3.2 and 3.3, A is a locally connected planar continuum which contains no simple closed curve as a retract. It follows that A cannot separate $\mathbb{R}^2$, and is therefore an absolute retract [2, V;13.1]. □

The theorem in 1.8 has now been proved by 1.6, 3.4, and 3.5.

4. Examples of ε-MAFS-spaces that are not AFS-spaces

Definition 4.1. Let X be a continuum. We define a metric, ρ, for X to be an ε-admissible metric for X provided that there is a homotopy, $h : C(X) \times [0,1] \to C(X)$ such that the following conditions are satisfied for all $A \in C(X)$:

(1) $h(A,0) = A$;
(2) $\operatorname{diam}[h(A,t)] < \operatorname{diam}[h(A,s)]$, whenever $0 \leqslant s < t$ and $\operatorname{diam}[h(A,s)] > 0$;
(3) $\operatorname{diam}[h(A,1)] \leqslant \varepsilon$.

Lemma 4.2. *Let X be a continuum with an ε-admissible metric, ρ, and let $h : C(X) \times [0,1] \to C(X)$ be a homotopy satisfying the conditions in* 4.1. *Then, for any $\eta \geqslant \varepsilon$ with $\eta > 0$, the function $\tau : C(X) \to [0,1]$ given by*

$$\tau(A) = \min\bigl\{t \in [0,1]\colon\ \operatorname{diam}_\rho\bigl[h(A,t)\bigr] \leqslant \eta\bigr\}$$

is continuous.

Proof. Since $\eta \geqslant \varepsilon$, it is clear that τ is well-defined by 4.1(3). Let $A \in C(X)$, and let $\{A_k\}_{k=1}^{\infty}$ be a sequence in $C(X)$ with $A_k \to A$. Observe that since $[0,1]$ is sequentially compact, we have that $\{\tau(A_k)\}_{k=1}^{\infty}$ has a convergent subsequence. To see that τ is continuous, it is enough to show that

($*$) every convergent subsequence of $\{\tau(A_k)\}_{k=1}^{\infty}$ converges to $\tau(A)$.

Let $\{\tau(A_{k_i})\}_{k=1}^{\infty}$ be a convergent subsequence of $\{\tau(A_k)\}_{k=1}^{\infty}$, and let $t_0 = \lim \tau(A_{k_i})$. Without loss of generality, we may assume that either $A_{k_i} \in C_\eta(X)$ for every $i \in \{1,2,\ldots\}$, or $A_{k_i} \in C(X) - C_\eta(X)$ for every $i \in \{1,2,\ldots\}$. Observe that if $A_{k_i} \in C_\eta(X)$ for every $i \in \{1,2,\ldots\}$, then ($*$) is clear from the observation that $\tau(B) = 0$ if and only if $B \in C_\eta(X)$. Otherwise, if $A_{k_i} \in C(X) - C_\eta(X)$ for every $i \in \{1,2,\ldots\}$, then the continuity of diam [10, 1.211] gives that $\operatorname{diam}[h(A_{k_i}, \tau(A_{k_i}))] = \eta$ for every $i \in \{1,2,\ldots\}$ by 4.1(2) and the definition of τ. Therefore, since $(A_{k_i}, \tau(A_{k_i}))$ converges to (A, t_0), we have that

$\operatorname{diam}[h(A,t_0)] = \eta$; so, $t_0 \geqslant \tau(A)$ by the minimality of $\tau(A)$. However, if $\tau(A) < t_0$, then we would have $\eta = \operatorname{diam}[h(A,t_0)] < \operatorname{diam}[h(A,\tau(A))] \leqslant \eta$ by 4.1(2) and the definition of τ. Therefore, $t_0 = \tau(A)$, giving $(*)$. This proves the lemma. □

Proposition 4.3. *If X is a locally connected continuum with an ε-admissible metric, ρ, then $C_{\rho,\eta}(X)$ is an* ANR*-space for all $\eta \geqslant \varepsilon$ with $\eta > 0$.*

Proof. Since X is a locally connected continuum, we have by [10, 1.96] that

$$C(X) \in \mathrm{AR}. \tag{21}$$

Therefore, we need only to consider the case when $0 < \eta < \operatorname{diam}_\rho(X)$. Let α be any real number for which $\eta < \alpha < \operatorname{diam}(X)$. Define $C_{<\alpha}(X)$ by

$$C_{<\alpha}(X) = \{A \in C(X)\colon \operatorname{diam}(A) < \alpha\}.$$

Clearly, $C_{<\alpha}(X)$ is an open subset of $C(X)$. Therefore, (21) and [2, IV; 10.1] give us that

$$C_{<\alpha}(X) \in \mathrm{ANR}(\mathcal{M}). \tag{22}$$

By assumption, there is a homotopy, $h : C(X) \times [0,1] \to C(X)$, that satisfies the conditions in 4.1. Let τ be the map described in 4.2, and observe that

$$\tau(A) = 0 \quad \text{for every } A \in C_\eta(X). \tag{23}$$

Now, define $r : C_{<\alpha}(X) \to C_\eta(X)$ by

$$r(A) = h\big(A, \tau(A)\big) \quad \text{for every } A \in C_{<\alpha}(X)$$

and observe that r is continuous by 4.2. It follows by (23) and 4.1(2). that r is a retraction of $C_{<\alpha}(X)$ onto $C_\eta(X)$. Therefore, $C_\eta(X) \in \mathrm{ANR}$ by (22) and [2, IV; 3.2]. This proves 4.3. □

The following notation will be used in our next result: For a compactum, Z, let Z' denote a copy of Z that is embedded in the face of the Hilbert cube, Q, given by $\{(0, x_2, x_3, \ldots)\colon 0 \leqslant x_i \leqslant \frac{1}{i}, i = 2, 3, \ldots\}$, and let $v = (1, \frac{1}{2}, \frac{1}{3}, \ldots) \in Q$. Let $\Lambda(Z)$ denote the geometric cone over Z, defined explicitly by

$$\Lambda(Z) = \{(1-t)p + tv\colon\ p \in Z',\ t \in [0,1]\}.$$

Proposition 4.4. *If Z is a locally connected compactum, then there is a metric on $\Lambda(Z)$ for which $C_\eta(\Lambda(Z))$ is an* AR*-space for every $\eta > 0$.*

Proof. Let ρ denote the subspace metric on $\Lambda(Z)$ inherited from Q, and let $h : \Lambda(Z) \times [0,1] \to \Lambda(Z)$ be the natural homotopy on $\Lambda(Z)$ which contracts $\Lambda(Z)$ to $\{v\}$; specifically, let $h(p,t) = (1-t)p + tv$ for all $(p,t) \in \Lambda(Z) \times [0,1]$. Let $H : C(\Lambda(Z)) \times [0,1] \to C(\Lambda(Z))$ denote the homotopy on $C(\Lambda(Z))$ induced by h; that is, $H(A,t) = \bigcup\{h(p,t)\colon\ p \in A\}$. It is easily seen that H satisfies 4.1(1)–(3) for $\varepsilon = 0$. Hence, $\Lambda(Z)$ is a locally connected continuum [13, 27.13], [13, 27.12] with a 0-admissible metric; therefore, 4.3 gives us that $C_\eta(\Lambda(Z)) \in \mathrm{ANR}$ for all $\eta > 0$. Furthermore, since $C_\eta(\Lambda(Z))$ is contractible and compact, it follows from [2, IV; 9.1] that $C_\eta(\Lambda(Z)) \in \mathrm{AR}$ for all $\eta > 0$. This proves the proposition. □

Proposition 4.5. *Let A be a compactum. If there exists a metric, d, on A such that, for every $\delta > 0$ there is some $0 \leqslant \eta \leqslant \delta$ for which $C_{d,\eta}(A)$ is an* AR*-space, then $A \in \varepsilon$*-MAFS.

Proof. Let Z be a compactum which contains a copy, A', of A. Let h be a homeomorphism from A onto A', and let ρ be a metric on Z for which [1, Theorem 5]

$$\rho\big(h(x), h(y)\big) = d(x, y) \quad \text{for all } x, y \in A. \tag{$*$}$$

Let $\delta > 0$, and use $(*)$ to choose $0 \leqslant \eta \leqslant \delta$ such that $C_\eta(A') \in$ AR. Let $G : A' \to C_\eta(A')$ be the map given by $G(a) = \{a\}$ for all $a \in A'$. Then, since $C_\eta(A') \in$ AR by hypothesis, G can be extended to a map $F : Z \to C_\eta(A')$. Since $\eta \leqslant \delta$ and $\mathcal{F}(F) = A'$, it follows from 1.3 that $A \in \varepsilon$-MAFS, as desired. □

Corollary 4.6. *If A is a locally connected compactum, then $\Lambda(A) \in \varepsilon$*-MAFS.

Proof. This result follows immediately from 4.4 and 4.5. □

Corollary 4.7. *Let A be a continuum. Then A is a Peano continuum if and only if $\Lambda(A)$ is an ε*-MAFS-*space.*

Proof. If $\Lambda(A) \in \varepsilon$-MAFS, then $\Lambda(A)$ is locally connected by 2.2. Let v denote the vertex of $\Lambda(A)$. Then, $A - \{v\}$ is an open—hence, locally connected—subset of $\Lambda(A)$. So, since $A - \{v\}$ is homeomorphic to $A \times [0, 1)$, it follows that A is also locally connected [13, 27.13]. Hence, A is a Peano continuum. The converse is proved in 4.6. This proves 4.7. □

Theorem 4.8. *For every $n \in \{2, 3, \ldots, \infty\}$, there exists an n-dimensional ε*-MAFS-*space that does not belong to* AFS.

Proof. Borsuk describes a locally connected, n-dimensional continuum, X_n, for every $n \in \{1, 2, \ldots\}$, such that X_n satisfies $\mathbb{LC}^{n-1}$, but not $\mathbb{LC}^n$ [2, p. 30]. It follows by [2, I; 17.2] that X_n is not locally contractible. Hence, $\Lambda(X_n)$ is not locally contractible, and therefore fails to belong to AFS [7, Theorem 2]. However, $\Lambda(X_n) \in \varepsilon$-MAFS by 4.6. Since $\Lambda(X_n)$ has dimension $n + 1$, this proves 4.8 for each $n \in \{2, 3, \ldots\}$. Now, let X denote Borsuk's example of an infinite-dimensional compactum which is locally contractible but not an ANR-space [2, p. 125]. Then, since local contractibility implies local (arcwise) connectedness [2, p. 30], it follows from 4.6 that $\Lambda(X) \in \varepsilon$-MAFS. However, Martin shows that $\Lambda(X)$ is not an AFS-space [6, p. 43]. This completes the proof of 4.8. □

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